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INTERVAL DIFFERENTIAL EQUATIONS

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I. INTRODUCTION

As in Markov (1979) we shall be concerned with the interval space $\langle I(R),+,-,\bullet\rangle$, where $I(R)$ is the set of all compact intervals on the real line R. This space is an extension of the familiar space $\langle I(R),+,\bullet\rangle$ with operations

$$A + B = [a_1,a_2] + [b_1,b_2] = [a_1 + b_1,a_2 + b_2] \quad (A,B \in I(R)),$$

$$\alpha \bullet B \equiv \alpha B = [\min\{\alpha b_1,\alpha b_2\},\max\{\alpha b_1,\alpha b_2\}] \quad (\alpha \in R),$$

by means of the operation

$$A - B = [\min\{a_1 - b_1,a_2 - b_2\},\max\{a_1 - b_1,a_2 - b_2\}]. \quad \text{(S)}$$

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The space $\langle I(R),+,-,\bullet\rangle$ normed by means of the interval norm $|A| = |[a_1,a_2]| = \max\{|a_1|,|a_2|\}$ will be denoted by $\langle I(R),+,-,\bullet,|\cdot|\rangle$.

The normed interval space $\langle I(R),+,-,\bullet,|\cdot|\rangle$ is not linear, for the equalities $A - B = A + (-B)$ and $A + B = A - (-B)$ do not generally hold true. Nevertheless, $\langle I(R),+,-,\bullet,|\cdot|\rangle$ has a very interesting structure, quite convenient for practical purposes, that may be characterized as 'almost linear' or 'quasilinear'. This structure enables us to develop axiomatically various interval theories, and in particular, a calculus for interval functions.

In what follows we use the notations: $-B = (-1)B$, $A \ominus B = A + (-B)$, $A \oplus B = A - (-B)$, $w(A) = w([a_1,a_2]) = a_2 - a_1$, $[A \vee B] = [\min\{a_1,b_1\},\max\{a_2,b_2\}]$ for $A = [a_1,a_2]$, $B = [b_1,b_2] \in I(R)$ (and, in particular, $[\alpha \vee \beta] = [\min\{\alpha,\beta\},\max\{\alpha,\beta\}]$ for $\alpha,\beta \in R$).

The properties of $\langle I(R),+,-,\bullet\rangle$ involving the familiar operations $+$, $\bullet$ are well-known. We give a list of the basic properties of $\langle I(R),+,-,\bullet\rangle$ involving the operation (S).

For $A,B \in I(R)$, $\lambda \in R$,

i) $\lambda(A - B) = \lambda A - \lambda B$;

ii) $(-A) - B = (-B) - A$ (or $A \oplus B = B \oplus A$).

For λ, $\mu \in R$, $\lambda\mu \geq 0$,

iii) $(\lambda + \mu)A = \lambda A + \mu A$;

iv) $(\lambda - \mu)A = \lambda A - \mu A$.

For A, B, C, $D \in I(R)$ denote $\mu_1 = (w(A) - w(C))(w(B)-w(D))$

and $\mu_2 = (w(A) - w(B))(w(C) - w(D))$. Then,

$$\text{v)}\quad (A + B) - (C + D) = \begin{cases} (A - C) + (B - D), & \mu_1 \geq 0, \\ (A - C) \oplus (B - D), & \mu_1 < 0; \end{cases}$$

$$\text{vi)}\quad (A - B) + (C - D) = \begin{cases} (A + C) - (B + D), & \mu_2 \geq 0, \\ (A \oplus C) - (B \oplus D), & \mu_2 < 0,\ \mu_1 \geq 0, \\ (A \oplus C) \ominus (B \oplus D), & \mu_2 < 0,\ \mu_1 < 0; \end{cases}$$

$$\text{vii)}\quad (A - B) - (C - D) = \begin{cases} (A - C) - (B - D), & \mu_2 \geq 0,\ \mu_1 \geq 0, \\ (A - C) \ominus (B - D), & \mu_2 \geq 0,\ \mu_1 < 0, \\ (A \ominus C) - (B \ominus D), & \mu_2 < 0. \end{cases}$$

On the basis of the space $<I(R), +, -, \bullet, |\cdot|>$ one can develop a calculus for interval functions of real variable (that is, mappings of the form $F : R \to I(R)$). The concepts of continuity, limits and derivatives of interval functions can be introduced in the familiar formal manner by means of the interval norm $|\cdot|$ and the operation (S). We thereby obtain the well-known concepts of continuity and limits (cf. Moore (1966)), and a simple and useful concept of derivative:

$$F'(t) = \lim_{h \to 0} \frac{F(t + h) - F(t)}{h} \,. \tag{D}$$

The operation (D) for differentiation is inverse to the familiar integration of interval functions, which is in accordance with the ideas of Moore (1966), Ratschek (1977) and many others.

Some properties of the derivative (D) are given in Markov

(1979). Here we continue our study of the properties of the derivative (D), discussing thereby the differential equation $X'(t) = F(t,X(t))$, where F is a continuous function of a real variable t and of an interval variable X.

We also consider a generalization of the concept of limit and the corresponding concept of derivative.

II. DIFFERENTIATION OF INTERVAL FUNCTION OF A REAL VARIABLE

The derivative of the interval function $F(t) = [f_1(t), f_2(t)]$ can be described by means of the derivatives of the boundary functions f_1 and f_2 as follows:

<u>Proposition 1</u>. The interval function $F(t) = [f_1(t), f_2(t)]$ is differentiable at t in the following two cases:

i) both f_1 and f_2 are differentiable at t, then:

$$F'(t) = [f_1'(t) \vee f_2'(t)];$$

ii) the one-sided derivatives of f_1 and f_2 exist at t and

$$f_1'(t - 0) = f_2'(t + 0)$$

$$f_1'(t + 0) = f_2'(t - 0),$$

then

$$F'(t) = [f_1'(t-0) \vee f_1'(t+0)] = [f_2'(t-0) \vee f_2'(t+0)].$$

An important characteristic of the interval function $F = [f_1, f_2]$ is the width-function $w(F)$ defined by $w(F;t) = w(F(t)) = f_2(t) - f_1(t)$. If $w(F)$ is monotone in some interval $[\alpha, \beta]$, we say that F is w-monotonic in $[\alpha, \beta]$. We shall briefly say that F is w-monotonic at t if F is w-monotonic in a neighbourhood of t.

Example. The interval polynomial

$$P(t) = A_0 + A_1(t - t_0) + A_2(t - t_0)^2 + \dots + A_n(t - t_0)^n$$
$$\equiv \sum_{i=0}^{n} A_i(t - t_0)^i \tag{1}$$

is w-monotonic at every $t \neq t_0$. (1) is w-decreasing in $(-\infty, t_0]$ and w-increasing in $[t_0, \infty)$. The boundary functions p_1, p_2, defined by $P(t) = [p_1(t), p_2(t)]$ are differentiable in $(-\infty, t_0)$ and (t_0, ∞), and the derivative of P can be calculated for $t \neq t_0$ according to Proposition 1, case i). At $t = t_0$ the function (1) is not w-monotonic. However, the derivative $P'(t_0)$ can be calculated be means of Proposition 1, case ii). It is easy to obtain that

$$P'(t) = A_1 + 2A_2(t - t_0) + \dots + nA_n(t - t_0)^{n-1},$$

$$P''(t) = 2A_2 + 6A_3(t - t_0) + \dots + n(n - 1)A_n(t - t_0)^{n-2},$$

$$\dots$$

$$P^{(n)}(t) = n!\, A_n,$$

$$P^{(n+1)}(t) = 0,$$

for every $t \in R$.

The derivatives $P^{(k)}$, $k = 1,2,\dots,$ are interval polynomials of the form (1), that are w-decreasing in $(-\infty,t_0)$, and w-incressing in (t_0,∞). This can be briefly expressed by writing $w'(P^{(k)};t) < 0$ for $t < t_0$, $w'(P^{(k)};t) > 0$ for $t > t_0$, where $w'(P^{(k)};t)$ is the derivative of $w(P^{(k)})$ at t.

Clearly, differentiability of F at t does not necessarily imply differentiability of $w(F)$ at t; $F'(t)$ may exist but $w'(F;t)$ may not (cf. Proposition 1, case ii)). However, if F is differentiable and w-monotonic at t, then $w(F)$ is also differentiable at t. In this case instead of saying that F is w-increasing in $[\alpha,\beta]$ we may write $w'(F;t) \geq 0$, $t \in [\alpha,\beta]$, etc.

Proposition 2. The interval functions F and G are differentiable and w-monotonic at t and the real function g is differentiable at t. Then

$$(F(t) + G(t))' = \begin{cases} F'(t) + G'(t), & \text{if } w'(F;t)w'(G;t) \geq 0, \\ F'(t) \oplus G'(t), & \text{if } w'(F;t)w'(G;t) < 0; \end{cases}$$

$$(F(t) - G(t))' = \begin{cases} F'(t) - G'(t), & \text{if } w'(F;t)w'(G;t) \geq 0, \\ F'(t) \oplus G'(t), & \text{if } w'(F;t)w'(G;t) < 0; \end{cases}$$

$$(F(t)g(t))' = \begin{cases} F'(t)g(t) + F(t)g'(t), & \text{if } w'(F;t)g(t)g'(t) \geq 0, \\ F'(t)g(t) \oplus F(t)g'(t), & \text{if } w'(F;t)g(t)g'(t) < 0. \end{cases}$$

Proposition 3. If the interval function F is differentiable at u_0 and the real function φ is differentiable at t_0, where $u_0 = \varphi(t_0)$ then the composite function $G : G(t) = F(\varphi(t))$ is differentiable at t_0 and $G'(t_0) = F'(u_0)\varphi'(t_0)$.

Proposition 4. If F and G are differentiable and w-monotonic in $[\alpha,\beta]$ and $F'(t) \equiv G'(t)$ in $[\alpha,\beta]$, then there exists an interval constant $C \in I(R)$ such that

$$F(t) - G(t) = C, \text{ if } w'(F;t)w'(G;t) \geq 0;$$

$$F(t) \ominus G(t) = C, \text{ if } w'(F;t)w'(G;t) \leq 0.$$

III. INTEGRATION AND DIFFERENTIATION

The familiar definition of a Riemann-integrable interval function is: $F = [f_1, f_2]$ is R-integrable over an interval $\mathbf{D} = [\alpha \vee \beta]$ (symbolically: $F \in \mathbb{R}(\mathbf{D})$), if f_1 and f_2 are Riemann-integrable over $\mathbf{D}$. The Riemann integral of $F \in \mathbb{R}([\alpha \vee \beta])$ over $[\alpha \vee \beta]$ is defined by

$$\int_\alpha^\beta F(t)dt = \begin{cases} \left[\int_\alpha^\beta f_1(t)dt, \int_\alpha^\beta f_2(t)dt\right], & \text{if } \alpha \leq \beta \\ \left[\int_\alpha^\beta f_2(t)dt, \int_\alpha^\beta f_1(t)dt\right], & \text{if } \alpha > \beta. \end{cases} \tag{2}$$

The relation between the derivative (D) and the integral (2) is given in the following two propositions:

Proposition 5. If F is w-monotonic on $D = [\alpha \vee \beta]$ and $F' \in \mathbb{R}(\mathbf{D})$, then

$$\int_\alpha^\beta F'(t)dt = F(\beta) - F(\alpha).$$

Proposition 6. If F is a continuous interval function over $[\alpha \vee \beta]$, then $\int_{\alpha}^{\beta} F(s)ds$ is a differentiable interval function at $t \in [\alpha \vee \beta]$ and

$$\frac{d}{dt}\left(\int_{\alpha}^{t} F(s)ds\right) = F(t),\ t \in [\alpha \vee \beta].$$

According to the above proposition the function

$$G(t) = \int_{\alpha}^{t} F(s)ds \tag{3}$$

is differentiable at every $t \in [\alpha \vee \beta]$. Assume now that the interval function $G(t)$ is defined in a neighbourhood of α by means of (3). Since $G'(\alpha + 0) = G'(\alpha - 0) = F(\alpha)$, we conclude that G is differentiable at α (however, $w(G;t)$ may not be differentiable at $t=\alpha$). By means of (2) we observe that $G(t)$ is w-decreasing in $[\alpha - \varepsilon, \alpha]$ and is w-increasing in $[\alpha, \alpha + \varepsilon]$, $\varepsilon > 0$. Thus the interval function $G(t)$ defined by (3) for $t \in [\alpha - \varepsilon, \alpha + \varepsilon]$ has the following two properties:

1) $G(t)$ is differentiable in $[\alpha - \varepsilon, \alpha + \varepsilon]$;
2) $G(t)$ is w-decreasing in $[\alpha - \varepsilon, \alpha]$ and w-increasing in $[\alpha, \alpha + \varepsilon]$.

We shall briefly say that the interval function G is $><$-like at α, if G is defined in a neighbourhood of α and satisfies 1) and 2). As an example of a $><$-like at t_0 function we may point out the interval polynomial (1) and its derivatives P', P'',... . If G is $><$-like at α then $G + C$, $C = \text{const} \in I(R)$, is also $><$-like at α. However, $G - C$ may either be $><$-like at α, or $<>$-like at α, that is:

1) G is differentiable in $[\alpha - \varepsilon, \alpha + \varepsilon]$,

2) G is w-increasing in $[\alpha - \varepsilon, \alpha]$ and w-decreasing in $[\alpha, \alpha + \varepsilon]$.

Proposition 7 (Integration by parts). If F is continuous in $D = [\alpha \vee \beta]$ and g is a continuous real function in D and such that sign $g(t)$ = const, $t \in D$, then

$$\int_\alpha^\beta F(t)dt \int_\alpha^\beta g(t)dt = \int_\alpha^\beta F(t)\left(\int_\alpha^t g(s)ds\right)dt + \int_\alpha^\beta g(t)\left(\int_\alpha^t F(s)ds\right)dt.$$

Another integration-by-parts-proposition is the following:

Proposition 8. If F is differentiable and w-monotonic in $D = [\alpha \vee \beta]$ and g is differentiable in D and such that $w'(F;t)g(t)g'(t) \geq 0$ for $t \in D$, then

$$F(t)g(t)\Big|_\alpha^\beta = \int_\alpha^\beta F'(t)g(t)dt + \int_\alpha^\beta F(t)g'(t)dt.$$

Remark. In case that $w'(F;t)g(t)g'(t) < 0$ we may write

$$F(t)g(t)\Big|_\alpha^\beta = \int_\alpha^\beta F'(t)g(t)dt \ominus \int_\alpha^\beta F(t)g'(t)dt$$

by the additional assumption that sign $\{w(F'(t)g(t)) - w(F(t)g'(t))\}$ = const for $t \in D$.

Proposition 9 (Change of variable in an interval integral). If $F \in \mathbb{R}([a,b])$ and $g(t)$ is an increasing differentiable function defined over $[\alpha,\beta]$, such that $g' \in \mathbb{R}([\alpha,\beta])$ and $g(\alpha) = a$, $g(\beta) = b$, then

$$\int_a^b F(t)dt = \int_\alpha^\beta F(g(t))g'(t)dt.$$

Remark. Propositions 7 and 9 do not involve interval derivatives and therefore are possibly well-known.

The Taylor's theorem takes its familiar form for interval functions F, such that $F, F', F'', \ldots, F^{(n)}$ are ><-like at a point t_0.

Proposition 10. The interval function F is n + 1 time differentiable in $[t \vee t_0]$ and $F^{(n+1)} \in \mathbb{R}([t \vee t_0])$. If the functions

$$F, F', \ldots, F^{(n)} \text{ are } \begin{Bmatrix} \text{w-increasing in } [t_0, t] \text{ in case that } t > t_0 \\ \text{w-decreasing in } [t, t_0] \text{ in case that } t < t_0 \end{Bmatrix},$$

then

$$F(t) = F(t_0) + F'(t_0)(t - t_0) + F''(t_0)(t - t_0)^2/2! + \ldots$$

$$+ F^{(n)}(t_0)(t - t_0)^n/n! + \frac{1}{n!} \int_{t_0}^{t} F^{(n+1)}(s)(t - s)ds.$$

A similar proposition can be formulated for interval functions F such that $F, F', \ldots, F^{(n)}$ are <>-like at t_0:

Proposition 11. F is n + 1 time differentiable and $F^{(n+1)}$ is R-integrable in $[t \vee t_0]$. If the functions

$$F, F', \ldots, F^{(n)} \text{ are } \begin{Bmatrix} \text{w-decreasing in } [t_0, t] \text{ if } t > t_0 \\ \text{w-increasing in } [t, t_0] \text{ if } t < t_0 \end{Bmatrix}, \text{ then}$$

$$F(t) = \Big(\sum_{i \in \Omega_0(n)} F^{(i)}(t_0)(t - t_0)^i/i! + R_0(n) \Big)$$

$$\oplus \Big(\sum_{i \in \Omega_1(n)} F^{(i)}(t_0)(t - t_0)^i/i! + R_1(n) \Big),$$

where $\Omega_0(n) = \{0,2,4,\dots,2k\}$, $n - 1 \le 2k \le n$; $\Omega_1(n) = \{1,3,\dots, 2\ell + 1\}$, $n - 1 \le 2\ell + 1 \le n$,

$$R_0(n) = \begin{cases} R(n) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}; \quad R_1(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ R(n) & \text{if } n \text{ is even} \end{cases};$$

$$R(n) = (1/n!) \int_{t_0}^{t} F^{(n+1)}(s)(t - s)ds.$$

IV. THE INTERVAL DIFFERENTIAL EQUATION $X' = F(t,X)$

Consider an interval-valued function $F(t,X)$ of two variables of different type: a real variable t and an interval variable X. We shall assume that t varies in some interval $[t_0,t+\alpha]$, $\alpha > 0$, and X takes values from some neighbourhood of X_0, that is a set of intervals of the form $\{X : [X - X_0] \le \beta,\ X \in I(R)\}$ where $\beta > 0$. The domain of $F = F(t,X)$ will be denoted by $\mathbb{D}$, i.e. $\mathbb{D} = \{(t,X) : t_0 \le t \le t_0 + \alpha,\ |X - X_0| \le \beta\}$. F is continuous in $\mathbb{D}$ if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$, such that $|F(t,X) - F(t_1,X_1)| < \varepsilon$ whenever $(t,X),(t_1,X_1) \in \mathbb{D}$, $|t - t_1|<\delta$, $|X - X_1| < \delta$. F satisfies a Lipschitz condition with respect to X on $\mathbb{D}$ if there is a real $k > 0$ such that

$$|F(t,X_1) - F(t,X_2)| \le k|X_1 - X_2|$$

for every $(t,X_1),(t,X_2) \in \mathbb{D}$.

Consider now the interval differential equation $X' = F(t,X)$ with initial condition $X(t_0) = X_0$, where X_0 is a given interval. We shall be looking for an interval $T = |t_0, t_1|$ and an interval function X, differentiable on T and such that: 1) $X(t_0) = X_0$; 2) $X'(t) = F(t, X(t))$ for every $t \in T$.

Example 1. Consider the problem:

$$X'(t) = C \quad (C \in I(R)) \tag{E1}$$
$$X(0) = 0.$$

There are infinitely many differentiable interval functions X, that are solutions of (E1) in the interval $T = [0,\infty)$. For instance, the function

$$X_p(t) = \begin{cases} Ct, & \text{for } 0 \le t \le p, \\ C(t - 2p), & \text{for } t > p \end{cases}$$

is a solution of (E1) for every choice of the positive parametre p.

This example shows that even the solution of the simple problem $X' = \mathrm{Const}$, $X(t_0) = X_0$ is not unique. However, there is a unique solution to (E1) that is w-increasing in $[0,\infty)$; this is the function $X_0(t) = Ct$.

Example 2. Consider the interval differential problem

$$X'(t) = F(t) \tag{E2}$$
$$X(t_0) = X_0.$$

Integrating $X'(s) = F(s)$ for $t_0 \le s \le t$ we obtain

$\int_{t_0}^{t} X'(s)ds = \int_{t_0}^{t} F(s)ds$. In order to 'solve' the integral in the left-hand side we have to require that X is w-monotonic in $[t_0,t]$. If $w(X_0) = 0$ then X can be only w-increasing in $[t_0,t]$. If $w(X_0) > 0$ then X can be also w-decreasing in some interval $[t_0,t_0 + \varepsilon]$. Assume that X is w-increasing. Then, according to <u>Proposition 5</u>, we have $X(t) - X(t_0) = \int_{t_0}^{t} F(s)ds$ and, since $w(X(t)) > w(X(t_0))$, we may write $X(t) = X(t_0) + \int_{t_0}^{t} F(s)ds$. Thus, by integrating (E2) and using <u>Proposition 5</u> we arrive to a w-increasing solution

$$X(t) = X_0 + \int_{t_0}^{t} F(s)ds.$$

Examples 1 and 2 show that it is natural to look for w-monotonic solutions and especially for w-increasing solutions in $[t_0,t_1]$ (when considering solutions in $[t_1,t_0]$ it is natural to want them to be w-decreasing). In most cases it is easy to obtain all solutions from a w-monotonic solution by a simple procedure.

We thus arrive to the following formulation of the problem:

<u>Problem</u>. Given are a real number t_0, an interval X_0 and an interval-valued function $F = F(t,X)$ of a real and an interval arguments, which is continuous in the set $\mathbb{D} = \{(t,X) : t_0 \le t \le t_0 + \alpha, |X - X_0| < \beta\}$ where α, β are some positive numbers. Find an interval $T = [t_0,t_0 + \alpha_1]$, $0 < \alpha_1 \le \alpha$ and an interval function X, differentiable on T and such that

i) $|X(t) - X_0| < \beta$ for $t \in T$;

ii) $X'(t) = F(t,X(t))$ for $t \in T$,

iii) $X(t_0) = X_0$,

iv) X is w-increasing on T.

In this section we shall briefly denote this problem by

$$X' = F(t,X) \qquad (4)$$
$$X(t_0) = X_0.$$

Proposition 12. If the interval T and the interval function X are solutions of (4) then X satisfies the interval integral equation

$$X(t) = X_0 + \int_{t_0}^{t} F(s,X(s))d\tau, \quad \text{for } t \in T, \qquad (5)$$

and conversely, if X is a continuous interval function, satisfying (5) in some interval T, then T and X are solutions to (4).

The following proposition gives a sufficient condition for existence and uniqueness of solution of (4).

Proposition 13. Given are $t_0 \in R$, $X_0 \in I(R)$, and two positive numbers α, β. If the interval function $F(t,X)$ is continuous in $\mathbb{D} = \{(t,X) : t_0 \le t \le t_0 + \alpha, |X - X_0| < \beta\}$ and $|F(t,X)| \le \mu$ in $\mathbb{D}$, then there exists a solution $X = X(t)$ of (4) in the interval $T = [t_0, t_0 + \min\{\alpha, \beta/\mu\}]$. If, in addition, F satisfies a Lipschitz condition with respect to X on $\mathbb{D}$, then this solution is unique.

Example 3. Consider the problem

$$\begin{aligned} X' &= 3X^{2/3} \\ X(0) &= 0 \end{aligned} \tag{E3}$$

where $X^{2/3} = \{x^{2/3} : x \in X\}$.

The function $F(t,X) = 3X^{2/3}$ in (E3) is not Lipschitzian with respect to X. There are infinitely many solutions of (E3). Three such solutions are $X_1(t) \equiv 0$, $X_2(t) = t^3$ and $X_3(t) = [0,1]t^3$ for $t \in [0,\infty)$.

Example 4. Consider the problem

$$\begin{aligned} X' &= a(t)X + B(t) \\ X(t_0) &= X_0, \end{aligned} \tag{E4}$$

where a(t) is a real-valued continuous function with constant sign on some interval T and B(t) is a continuous interval function defined on T.

The unique solution of (E4) on T is given by

$$X = ((X_1 + X_2) + (X_1 - X_2))/2,$$

where

$$X_1(t) = \varphi(t)(X_0 + \int_{t_0}^{t} \varphi^{-1}(s)B(s)ds),$$

$$X_2(t) = \varphi^{-1}(t)(X_0 + \int_{t_0}^{t} \varphi(s)B(s)ds)$$

and $\varphi(t) = \exp(\int_{t_0}^{t} a(s)ds)$, $\varphi^{-1}(t) = 1/\varphi(t)$.

If $a(t) > 0$ then

$$X(t) = ((X_1(t) + X_2(t)) + (X_1(t) - X_2(t)))/2 \tag{6}$$

$$= X_1(t) = \varphi(t)(X_0 + \int_{t_0}^{t} \varphi^{-1}(s)B(s)ds).$$

Consider the problem

$$x' = a(t)x + b(t),\ x(t_0) = c, \tag{7}$$

where c varies in some interval C and $b(t)$ varies in some interval function $B(t)$ (i.e. $b(t) \in B(t)$ for $t \in T$). Denote the solution of (7) by $x(b,c;t)$. Then, using (6), we see that for $a(t) > 0$

$$\{x(b,c;t): b(t) \in B(t),\ c \in C\} = X(t)$$

where $X(t)$ is the solution of (E4).

Similar consideration may find place in some optimal control problems as the following example shows:

Example 5. Consider a control system described by the linear differential equation

$$x'(t) = a(t)x(t) + b(t)u(t) + c(t) \tag{E5}$$
$$x(t_0) = x_0,$$

where x and u are respectively the state and control variables; a, b, c are integrable in some interval T and $a(t) > 0$ in T. Let $U(t)$ be a continuous interval function on T. Define the set

Ω_t of admissible controls for $s \in |t_0,t|$ by

$$\Omega_t = \{u : u \text{ is measurable, } u(s) \in U(s) \text{ for } t_0 \le s \le t\}.$$

The attainable set for the problem (E5) is defined by

$$A(t) = \{x_n(t) : x_n \text{ satisfies (E5) for } u \in \Omega_t\}.$$

It is to be noted that $A(t)$ is an interval w-increasing function of t. Moreover, $A(t)$ is a solution of the interval differential problem

$$X'(t) = a(t)X(t) + b(t)U(t) + c(t)$$
$$X(t_0) = x_0.$$

In other words we obtain the equation for the attainable set simply by putting U instead of u in (E5).

V. EXTENDED SEGMENT ANALYSIS

Our considerations by now outline a theory that might be called extended interval analysis. This theory is an extension of the interval analysis as initiated by Moore (1966) by introducing the operation (S) for subtraction.

In an analogous way we may extend the segment analysis as proposed by Sendov (1980) by means of the subtraction (S). We thus obtain a very interesting theory that may be called extended segment analysis.

The main difference between the interval analysis and the segment analysis consists in the different concepts of limit. The segment limit (S-limit) of an interval function $F = [f_1, f_2]$, defined in a neighbourhood of t, at t is:

$$\operatorname*{Slim}_{\tau\to t} F(\tau) = [\underline{\lim}_{\tau\to t} f_1(\tau),\ \overline{\lim}_{\tau\to t} f_2(\tau)] \tag{8}$$

where, as usually,

$$\underline{\lim}_{\tau\to t} f_1(s) = \sup_{\delta>0} \inf_{0<|t-\tau|<\delta} f_1(\tau),$$

$$\overline{\lim}_{\tau\to t} f_2(s) = \inf_{\delta>0} \sup_{0<|t-\tau|<\delta} f_2(\tau).$$

The following proposition shows the natural relation between the operation 'v' and the concept of S-limit:

<u>Proposition 14</u>. If the interval functions F and G are defined in a neighbourhood of t, then

$$\operatorname*{Slim}_{\tau\to t} (F \vee G)(t) = \operatorname*{Slim}_{\tau\to t} F(t) \vee \operatorname*{Slim}_{\tau\to t} G(t).$$

An important class of interval functions, introduced and studied in the segment analysis, is the class of S-continuous functions. An interval function F is S-continuous in the interval D, if $\operatorname*{Slim}_{\tau\to t} F(\tau) \subset F(t)$ for all $t \in D$. This definition and definition (8) can be extended for interval function of several (real and/or interval) variables in the usual manner.

We define the S-derivative of an interval function F by means of the subtraction (S) and the S-limit (8):

Definition. If F is an interval function defined in a neighbourhood of t, then the S-derivative of F at t is the interval

$$F'(t) = \operatorname*{Slim}_{h \to 0} \frac{F(t+h) - F(t)}{h} . \tag{9}$$

Now we can consider the interval differential problem

$$X' = F(t,X), \quad X(t_0) = X_0, \tag{10}$$

where F is S-continuous in a neighbourhood $\mathbb{D}$ of (t_0, X_0), where $\mathbb{D} = \{(t,X) : t_0 \leq t \leq t_0 + \alpha, |X - X_0| < \beta\}$, and the derivative X' is taken in the sense of (9).

By a solution of (10) in $T = [t_0, t_0 + \alpha_1]$, $\alpha \geq \alpha_1$, we mean a continuous interval function $X = X(t)$ such that:

1) $|X(t) - X_0| < \beta$ for $t \in T$;
2) X is w-monotonic;
3) $X(t_0) = X_0$, and
4) $X'(t) = F(t,X(t))$ a.e. in T (X' defined by (9)).

Proposition 15. If F(t,X) is S-continuous, bounded and inclusion monotonic with respect to X in $\mathbb{D}$, then there exists an interval T and a continuous in T interval function X(t) that is a solution of (10) in T.

REFERENCES

Markov, S. (1979). Calculus for interval functions of a real variable. Computing 22, 325-337.

Moore, R. (1966). "Interval Analysis". Prentice-Hall, Englewood Cliffs, N. J.

Ratschek, H. (1977). Mittelwertsätze für Intervallfunktionen. Beiträge zur Numerischen Mathematik, 6, 133-144.

Sendov, Bl. (1980). Some topics of segment analysis. (This volume.)